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Schur complement preconditioners for the Navier–Stokes equations

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SUMMARY

Mixed finite element formulations of fluid flow problems lead to large systems of equations of saddle-point type for which iterative solution methods are mandatory for reasons of efficiency. A successful approach in the design of solution methods takes into account the structure of the problem; in particular, it is well-known that an efficient solution can be obtained if the associated Schur complement problem can be solved efficiently and robustly. In this work we present a preconditioner for the Schur complement for the Oseen problem which was introduced in Kay and Loghin (Technical Report 99/06, Oxford University Computing Laboratory, 1999). We show that the spectrum of the preconditioned system is independent of the mesh parameter; moreover, we demonstrate that the number of GMRES iterations grows like the square-root of the Reynolds number. We also present convergence results for the Schur complement of the Jacobian matrix for the Navier–Stokes operator which exhibit the same mesh independence property and similar growth with the Reynolds number. Copyright © 2002 John Wiley & Sons, Ltd.

1. THE SCHUR COMPLEMENT APPROACH: AN OVERVIEW

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary Γ. Consider the Navier–Stokes equations in primitive variables with the following boundary conditions:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega\times(0,T) \tag{1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T) \tag{1b}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}^*(\mathbf{x},t) \qquad \text{on } \Gamma_D\times(0,T) \tag{1c}$$

$$\mathbf{n}\cdot\boldsymbol{\sigma} = 0 \qquad \text{on } \Gamma_N\times(0,T) \tag{1d}$$

with initial condition $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ in Ω. Here $\mathbf{n}$ is the outward normal to Γ and $\boldsymbol{\sigma} = -p\mathbf{I} + 2\nu\varepsilon(\mathbf{u})$ is the Cauchy stress tensor, with $\varepsilon = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$, the symmetric part of the velocity gradient.

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A fully-implicit time discretization scheme (e.g. a θ-method) coupled with standard linearizations of (1) leads to problems of the form:

$$-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{b} + \theta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2a}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \tag{2b}$$

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } \Gamma_D \tag{2c}$$

$$\mathbf{n}\cdot\boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N \tag{2d}$$

where $\mathbf{b}$ is a divergence-free vector field which is the solution computed at the previous step. We note that the choice $\rho=0$ defines the Picard linearization, whereas $\rho=1$ corresponds to the Newton method. While the former is computationally cheap, the rate of convergence is linear and in practice the method seems to work for larger values of ν. On the other hand, Newton's method with a quadratic rate of convergence is more expensive and requires a good initial guess; however, the method works for regimes where $\nu \ll 1$.

In either case, a stabilized mixed finite element discretization of (2) leads to a system of linear equations of the form:

$$K\mathbf{x} = K\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_p\end{pmatrix} = \begin{pmatrix}F+\rho M & B_1^t\\ B_2 & -C\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_p\end{pmatrix} = \begin{pmatrix}\mathbf{f}_u\\ \mathbf{f}_p\end{pmatrix} \tag{3}$$

where $F+\rho M$ is a vector 'advection–diffusion–reaction' operator, B_1^t, B_2 are discrete gradient and divergence operators including stabilization terms and C is a stabilization matrix. We note here that stabilization is not necessary if ν is large and the choice of finite element spaces satisfies the well-known inf–sup condition of Babuška and Brezzi; in this case $B_1 = B_2$.

Since the size of K is usually large, we restrict our attention to iterative solution methods for system (3). In particular, we note here two major classes of solution methods:

(i) multigrid methods;
(ii) Krylov subspace methods.

Both methods have been shown to be successful solvers for various choices of discretizations of (1) as well as for various ranges of the viscosity parameter ν; we refer the reader to References [1, 2] for a comparison of some of these methods.

In this work we report on the performance of a preconditioning technique employed in conjunction with a Krylov method. However, the resulting preconditioner could also be used in certain multigrid iterations, e.g. Reference [3], which employ a pressure solution method (see below).

In general, the solution of (3) is usually sought in two different ways:

- A global approach, where the solution $\mathbf{x}$ is computed iteratively.

A Krylov subspace approach employs an iterative solver such as GMRES or BiCGStab with right or left preconditioners given, respectively, by

$$P_\mathrm{R} = \begin{pmatrix}F+\rho M & B_1^t\\ O & -\hat{S}\end{pmatrix}, \qquad P_\mathrm{L} = \begin{pmatrix}F+\rho M & O\\ B_2 & -\hat{S}\end{pmatrix} \tag{4}$$

where $\hat{S}$ is an approximation to the *pressure Schur complement* $S = C + B_2(F + \rho M)^{-1}B_1^t$. We note here that if $\hat{S} = S$ convergence is guaranteed in at most 3 iterations [4].

- A pressure solution method, where $\mathbf{x}_u$ is eliminated from (3) and then the solution $\mathbf{x}_p$ is computed iteratively; $\mathbf{x}_u$ is then found in terms of $\mathbf{x}_p$.

This approach leads to a system for $\mathbf{x}_p$ of the form:

$$S\mathbf{x}_p = B_2(F + \rho M)^{-1}\mathbf{f}_u - \mathbf{f}_p \tag{5}$$

which, when solved iteratively, also needs an approximation to S.

It is clear that both approaches need to approximate (i) the Schur complement S and (ii) the vector 'advection–diffusion–reaction' operator $F + \rho M$. Assuming the latter task can be effectively achieved for $\rho = 0$ we present a useful approximation of the Schur complement S for this case. The resulting preconditioner will then be tested for the case $\rho = 1$. In the following $\rho = 0$ unless otherwise stated.

We first note that for the Stokes problem a useful approximation of the Schur complement was introduced in Reference [5]

$$\hat{S} = (\nu M_p^{-1} + \theta A_p^{-1})^{-1} \tag{6}$$

where M_p and A_p are the projections of the identity and a Neumann Laplacian onto the pressure space. We note also that for steady-state Stokes (6) becomes $\hat{S} = M_p/\nu$, which was analysed in References [6, 7].

Naturally, the above choices of $\hat{S}$ were considered for the Navier–Stokes equations and in particular for the Oseen problem which corresponds to (2) with $\rho = 0$. Results are reported in Reference [8] for the time-dependent problem and in References [7, 9] for steady-state problems and for *stable* formulations. We also note here the multigrid approach in Reference [3] which requires an approximation of the Schur complement; the approach in Reference [3] uses among other choices the preconditioner (6) to solve the pressure Schur complement problem (5). Analytic and numerical results in the above references show that the preconditioned system has a spectrum independent of the mesh parameter and conclude that convergence is mesh-independent. However, the viscosity parameter ν is also an important parameter and it is desirable that convergence be independent of or mildly dependent on ν as $\nu \to 0$. For example, the choice (6) with $\theta = 0$ yields a number of iterations which was shown to increase linearly with $1/\nu$ [7]; moreover, numerical results in References [7, 9] seem to indicate that the preconditioner is useful for a limited range of ν. That the convergence rate deteriorates as $\nu \to 0$ is a somewhat expected result since the non-normality of S cannot be matched by symmetric preconditioner (6).

Alternative preconditioners which tried to deal with the non-symmetry inherent in the Schur complement were proposed in References [9, 10]. Elman suggested the approximation

$$(\hat{S})^{-1} = (BB^t)^{-1}BFB^t(BB^t)^{-1}$$

for stable formulations for which $B_1 = B_2 = B$. This choice reduces the dependence on ν to $\nu^{-1/2}$ but introduces an h^{-1} dependence for the number of iterations. Moreover, though efficiently applied to the MAC finite difference scheme, the implementation for finite elements requires a further efficient approximation for $(BB^t)^{-1}$.

The choice of preconditioner we present in this paper was introduced in Reference [10] and is given by

$$(\hat{S})^{-1} = M_p^{-1} F_p A_p^{-1} \tag{7}$$

with M_p, A_p defined as above and F_p the projection onto the pressure finite element space of the velocity operator of Equation (2a), $-\nu\Delta + \mathbf{b}\cdot\nabla + \theta$. Note that when $\mathbf{b}=0$ we recover the preconditioner (6). The numerical results presented in Reference [10] together with those in Reference [11] show no mesh dependence and a dependence on ν of order $\nu^{-1/2}$ or less. The analysis in Reference [12] confirms and refines these results. In this paper we review these results and demonstrate that the number of iterations is of order $O(R^{1/2})$ where R is the Reynolds number.

2. CONVERGENCE ANALYSIS

A mixed formulation of problem (2) with $\rho=0$ involves choosing appropriate spaces for the velocity and pressure

$$\mathbf{V} \subset [H_E^1(\Omega)]^2 = \{\phi \in [H^1(\Omega)]^2 \colon\ \phi\,|_{\Gamma_D} = 0\}, \quad P \subset L_0^2(\Omega) = \{p \in L^2(\Omega)\colon\ \langle p, 1\rangle = 0\}$$

and results in the following weak formulation:

$$\text{Given } \mathbf{f} \in [L^2(\Omega)]^2, \text{ find } (\mathbf{u}, p) \in H = \mathbf{V}\times P \text{ such that}$$

$$B(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in H \tag{8}$$

where

$$B(\mathbf{w}, r; \mathbf{v}, q) = \nu\langle\nabla\mathbf{w}, \nabla\mathbf{v}\rangle + \langle\mathbf{b}\cdot\nabla\mathbf{w} + \theta\mathbf{w}, \mathbf{v}\rangle - \langle r, \operatorname{div}\mathbf{v}\rangle - \langle q, \operatorname{div}\mathbf{w}\rangle, \quad F(\mathbf{v}, q) = \langle\mathbf{f}, \mathbf{v}\rangle \tag{9}$$

Existence and uniqueness are guaranteed provided the bilinear form $B(\cdot,\cdot,\cdot,\cdot)$ is coercive and continuous with respect to some suitable norm on H. Various discrete formulations which satisfy these requirements can be found in the literature; we employ here the stabilized formulation of Franca and Frey [13]

$$\text{Find } (\mathbf{u}, p) \in \mathbf{V}^h \times P^h = H^h \subset H \text{ such that}$$

$$B_\delta(\mathbf{u}, p; \mathbf{v}, q) = F_\delta(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in H^h \tag{10}$$

where

$$B_\delta(\mathbf{w}, r; \mathbf{v}, q) = B(\mathbf{w}, r; \mathbf{v}, q) + \beta\langle\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{v}\rangle$$
$$\qquad + \sum_{T\in\mathscr{T}^h} \delta_T\langle -\nu\Delta\mathbf{w} + \mathbf{b}\cdot\nabla\mathbf{w} + \theta\mathbf{w} + \nabla r, \nu\Delta\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{v} - \nabla q\rangle_T$$

$$F_\delta(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + \sum_{T\in\mathscr{T}^h} \delta_T\langle\mathbf{f}, \nu\Delta\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{v} - \nabla q\rangle_T$$

Here $\delta_T = O(h_T^2/v)$ is a mesh function defined on the computational domain $\Omega_h = \bigcup T$, where T are simplices of diameter h_T. The above choice of finite element spaces produces the matrix problem (3) with $\rho = 0$ and the blocks defined via

$$\langle F\mathbf{w}, \mathbf{v}\rangle = a(\mathbf{w}, \mathbf{v}) = v\langle \nabla\mathbf{w}, \nabla\mathbf{v}\rangle + \langle \mathbf{b}\cdot\nabla\mathbf{w}, \mathbf{v}\rangle + \theta\langle \mathbf{w}, \mathbf{v}\rangle + \beta\langle \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}\rangle$$
$$+ \sum_{T\in\mathscr{T}^h} \delta_T\langle -v\Delta\mathbf{w} + \mathbf{b}\cdot\nabla\mathbf{w} + \theta\mathbf{w}, v\Delta\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{v}\rangle_T \tag{11}$$

$$\langle B_1\mathbf{v}, r\rangle = -\langle \operatorname{div}\mathbf{v}, r\rangle - \sum_{T\in\mathscr{T}^h} \delta_T\langle -v\Delta\mathbf{v} - \mathbf{b}\cdot\nabla\mathbf{v}, \nabla r\rangle_T \tag{12}$$

$$\langle B_2\mathbf{w}, q\rangle = -\langle \operatorname{div}\mathbf{w}, q\rangle - \sum_{T\in\mathscr{T}^h} \delta_T\langle -v\Delta\mathbf{w} + \mathbf{b}\cdot\nabla\mathbf{w} + \theta\mathbf{w}, \nabla q\rangle_T \tag{13}$$

$$\langle Cr, q\rangle = -\sum_{T\in\mathscr{T}^h} \delta_T\langle \nabla r, \nabla q\rangle_T \tag{14}$$

for $\mathbf{w}, \mathbf{v}\in\mathbf{V}^h$, $r, q\in P^h$. For the above choice of finite element spaces our preconditioner is defined as in (7) with A_p, F_p, M_p defined via

$$\langle F_p q, r\rangle = a(q, r), \quad \langle M_p q, r\rangle = \langle q, r\rangle, \quad \langle A_p q, r\rangle = \langle \nabla q, \nabla r\rangle, \;\; q, r\in P^h$$

The following result which can be found in Reference [12] provides mesh-independent bounds on the spectrum of the preconditioned Schur complement for the Oseen problem.

Theorem 2.1

Let $S = C + B_2F^{-1}B_1^t$ denote the Schur complement associated with the Oseen problem and let $\hat{S}$ be defined as in (7), with A_p, F_p, M_p defined as above. Then there exist constants C_1, C_2 such that

$$C_1\,\frac{v(v+\theta)}{b^2} \leqslant |\lambda_i(S\hat{S}_p^{-1})| \leqslant C_2\,\frac{\theta + b}{v}$$

where $b = \|\mathbf{b}\|$.

Note that for steady problems ($\theta = 0$) the spectrum lies in an annulus in the complex plane with outer radius $R = \|\mathbf{b}\|/v$ and inner radius R^{-2}.

One can use this result to infer the mesh-independence of iterative methods such as GMRES. The residuals $\mathbf{r}^k$ in the GMRES iteration applied to (5) satisfy [14]

$$\frac{\|\mathbf{r}^k\|}{\|\mathbf{r}^0\|} \leqslant \frac{\mathscr{L}(\Gamma_\varepsilon)}{2\pi\varepsilon} \min_{p_k(0)=1} \max_{z\in\Lambda_\varepsilon(S\hat{S}^{-1})} |p_k(z)| \tag{15}$$

where p_k denotes a polynomial of degree k and the set $\Lambda_\varepsilon(M) := \{z\in\mathbb{C}\colon \|(zI - M)^{-1}\| > \varepsilon^{-1}\}$ is the ε-pseudo-spectrum with contour length $\mathscr{L}(\Gamma_\varepsilon)$ of a matrix M. Given the bounds of Theorem 2.1, one would expect the right-hand side in the above inequality to be independent of the mesh parameter, i.e. that the pseudo-spectrum Λ_ε does not depend on the mesh parameter. Experiments suggest that this is indeed the case (see Reference [12]). Thus, for the pressure method (5), convergence of GMRES preconditioned with right preconditioner P_{R} defined in

(4) is mesh-independent. The result for the global method (3) preconditioned by P_R follows similarly after noting that $\Lambda(KP_R^{-1})=\Lambda(S\hat{S}^{-1})\cup\{1\}$.

The above results do not provide any insight into the convergence of GMRES with respect to the other parameters in the problem: $\varepsilon, \|\mathbf{b}\|, \theta$. For example, one could estimate Γ_ε to be of order R from the result of Theorem 2.1. However, the minimax problem in the above corollary does not have in general a known solution; see Reference [11] for a more detailed discussion. One can still get an insight into the convergence behaviour of GMRES: numerical experiments indicate that the number of iterations grows like $R^{1/2}$ (see next section) and seems to be virtually independent of the choice of θ. We expect to investigate this issue more closely in future work.

3. NUMERICAL RESULTS

In this section we consider the performance of the GMRES method applied to the global solution approach with right preconditioner P_R as in (4). We limit our experiments to the steady-state case, which seems to be the most trying for our preconditioning approach.

The factorization

$$P_R^{-1}=\begin{pmatrix}(F+\rho M)^{-1} & \\ & I\end{pmatrix}\begin{pmatrix}I & B_1^t\\ & -I\end{pmatrix}\begin{pmatrix}I & \\ & \hat{S}^{-1}\end{pmatrix}$$

indicates that we need to invert $F+\rho M$, A_p and M_p. However, these inverses do not have to be implemented exactly. The first is achieved with 5 iterations of GMRES with ILU preconditioning, the second with 3 iterations of the conjugate gradient method (CG) with ILU and the last with 3 iterations of CG with diagonal preconditioning. We note here that inverting F is potentially a computationally more intensive task: however, for the Picard iteration the matrix F is an advection–diffusion operator for which ILU appears to be a useful preconditioner.

We present results for three *steady* test problems, for the Q2Q1 finite element discretization:

(i) the regularized driven cavity;
(ii) flow in a box;
(iii) flow past a backward facing step.

The first and third problems are standard; the second is flow in the unit box defined by the following data (cf. (1))

$$\Gamma_1=\{(x,y)\in\Gamma_x:\ 0.25<x<0.75\},\quad \Gamma_N=\{(x,y)\in\Gamma_x:\ 0.25<y<0.75\}$$

$$\mathbf{u}^*|_{\Gamma_1}=(0,-16x^2+16x-3),\quad \mathbf{u}^*|_{\Gamma_D\setminus\Gamma_1}=(0,0)$$

We employed preconditioner P_R for problem (3) for each of the cases $\rho=0$ and 1. Writing (1) as $\mathscr{F}(w)=0$, where $w=(\mathbf{u},p)$, the GMRES stopping criterion at the nth non-linear (outer) iteration was in each case $\|\mathbf{r}^k\|/\|\mathscr{F}(w^n)\|\leqslant 10^{-2}$. The stopping tolerance for the outer iteration was 10^{-6}.

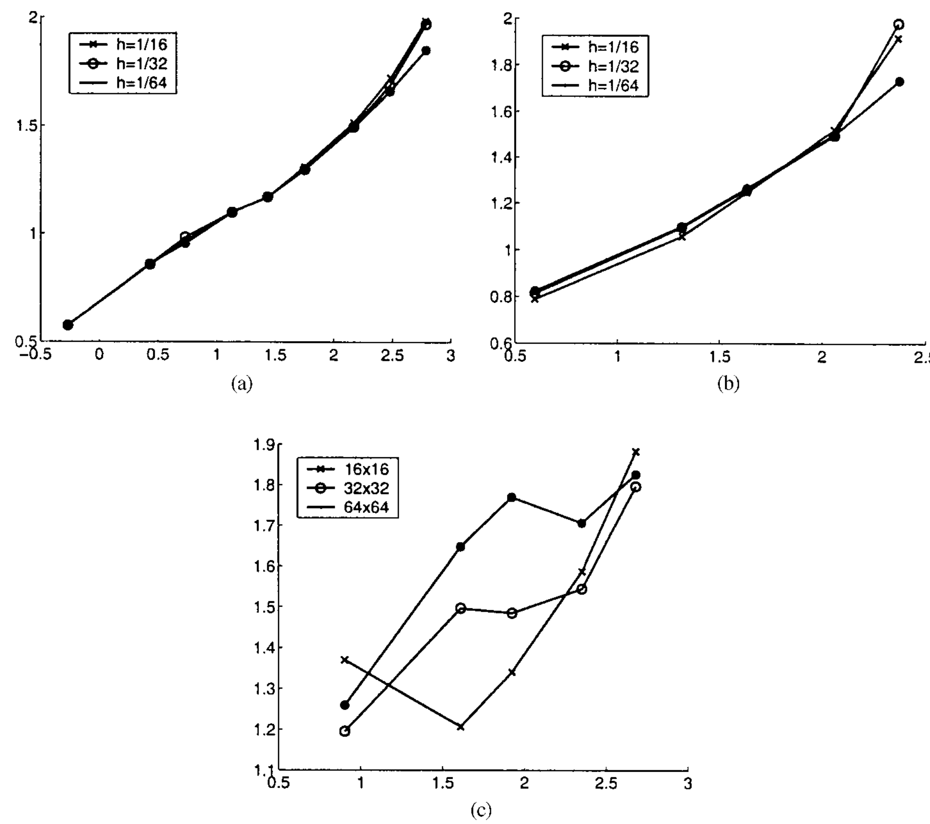

Figure 1. Loglog plot of the number of GMRES iterations versus $\|\mathbf{u}\|/\nu$: (a) Cavity problem, (b) box problem and (c) step problem.

3.1. *The Picard iteration:* $\rho = 0$

The results for the preconditioned Picard iteration are shown in Figures 1 and 2. The number of GMRES iterations shown represents the average over the number of non-linear (outer) iterations.

The performance of our preconditioner is indeed mesh-independent as predicted by Theorem 2.1 and the bound (15). Moreover, the dependence on the parameters seems to be completely described by the parameter $R = \|u\|/\nu$ which appears in the bounds of Theorem 2.1. More precisely, the numerics suggest that the number of iterations grows like $R^{1/2}$. We note here that for fixed ν, $\|\mathbf{u}\|$ varies from problem to problem. This suggests indeed that ν is not sufficient to characterize convergence of our iteration.

3.2. *The Newton iteration:* $\rho = 1$

The preconditioning technique described above was also successfully applied to the Jacobian matrix that arises in a Newton or Newton-type iteration. We chose to exhibit the dependence on R by implementing a continuation method, the Euler–Newton algorithm (see Reference

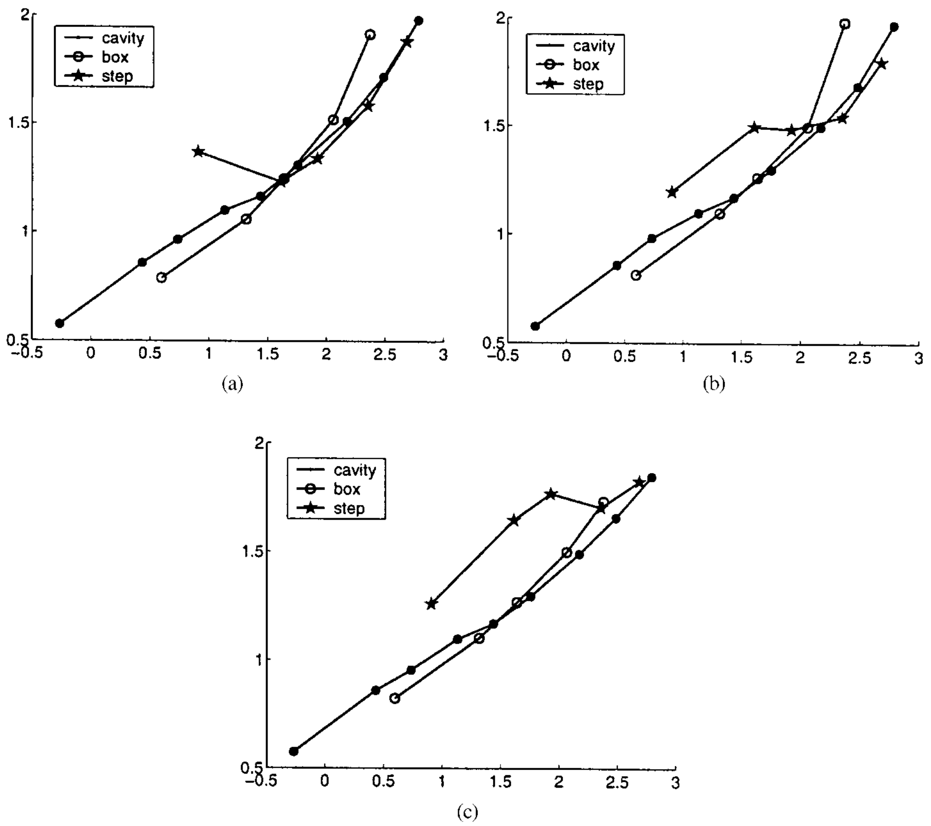

Figure 2. Loglog plot of the number of GMRES iterations versus $\|\mathbf{u}\|/\nu$: a comparison for each mesh: (a) $h=1/16$; (b) $h=1/32$; (c) $h=1/64$.

[12] for details). This is essentially a predictor–corrector method stepping forward in ν^{-1} with the Newton method acting as a corrector. The results are shown in Figure 3. Again we see the same mesh-independence, and roughly the same dependence on R. The jaggedness of the plots is due to our stepping strategy and choice of tolerances for the predictor (10^{-6}), corrector (10^{-3}) and GMRES algorithm. However, we do not expect to see an improvement with respect to R should these parameters be altered.

4. CONCLUSION

We presented a preconditioning technique for the linear system arising from the discretization of the Navier–Stokes equations. The algorithm is robust with respect to method of discretization, mesh parameter, time-discretization (see Reference [12]). The implementation is simple and modular: the main building blocks are an advection–diffusion–reaction solver and a Neumann Laplacian solver, which are available with most commercial software packages. The convergence is analysed in References [11, 12] and it is believed to be well-understood for

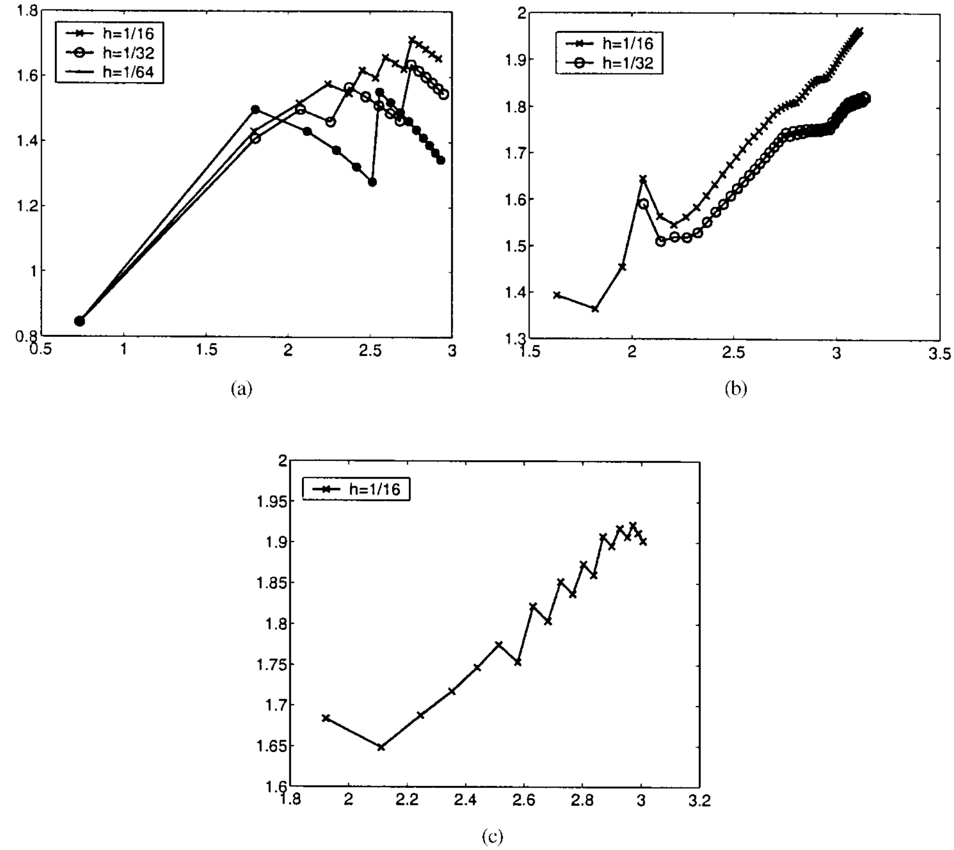

Figure 3. Loglog plot of the number of GMRES iterations versus $\|\mathbf{u}\|/\nu$: (a) Cavity problem, (b) box problem and (c) step problem.

the Picard method. We expect future work to validate the results presented for the Newton approach.

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